Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Journal of Sound and Vibration 274 (2004) 507–528
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

An inverse strategy for relocation of eigenfrequencies in structural design. Part II: second order approximate solutions

K. Farahani, H. Bahai*

Department of Systems Engineering, Brunel University, Uxbridge, Middlesex UB8 3PH, UK

Received 2 September 2002; accepted 27 May 2003

Abstract

This paper extends the first order formulations presented in Part I to second order methods for relocation of structural natural frequencies from their initial design values to new modified frequencies. The method is based on an inverse formulation and solution algorithm of the eigenvalue problem. Using the second order Taylor's expansion series, the required parameter variation to achieve a desired natural frequency shift for the structure is computed through second order differential or binomial equations. The proposed technique can also incorporate the design constraints or objective functions in the system equations. The formulations are quite generic and applicable to all finite element structures. The accuracy of the proposed methods is tested by conducting several case studies, the results of which demonstrate the validity of the technique for a wide range of practical problems.

1. Introduction

The first order approximation methods for modification of eigenvalues have been investigated in Part I of this work where good agreement between exact and approximate solutions was achieved. However, practical situations can still remain where larger or more accurate degrees of frequency shift are sought. For such cases, higher order derivative of eigenvalues and eigenvectors are required in order to establish second order differential equations or second order Taylor series expansion of eigenvalues.

The sensitivity of eigenvalues and eigenvectors are considered as important issues in structural design optimization, dynamic system identification and control. These sensitivities represent a linearized estimate of the change in the modal parameters, principally frequencies and modal

*Corresponding author. Tel.: +44-1895-274000x2609; fax: +44-1895-812556.
E-mail address: hamid.bahai@brunel.ac.uk (H. Bahai).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2003.11.013

shapes, due to perturbations of the stiffness and mass matrices of the numerical model. Various methods have been proposed in the past [1–5] for accurate and efficient computation of these sensitivity coefficients. As it will be seen, the accuracy of eigenvector derivatives depends on the number of eigenvectors taken into consideration. It is numerically uneconomical to compute many eigenvectors of a real structure with many degrees of freedom. Lin et al. [6] introduced a method to reduce the order of original finite element model [7], and a proposed practical perturbation method for computation of eigenvector derivatives which require only the eigenvectors themselves in order to considerably reduce the computational effort. Also, Alvin [8] has introduced an iterative procedure for computing eigenvector sensitivities due to finite element model parameter variations. His method is a preconditioned conjugate projected gradient-based technique and is intended to utilize the existing matrix factorizations developed for an iterative eigensolution such as Lanczos or Subspace Iteration.

2. Preliminaries

The first order methods for eigenvalue shift approximations have been investigated in part I of this work where good agreement between exact and approximate solutions was observed for small shifts of frequency. In situations where larger or more accurate frequency shifts are required, a higher order derivative of eigenvalues and eigenvectors are required in order to establish second order equations for modal variables. Consider again the eigenvalue problem free vibration of a finite element structure without damping:

$$\{[\mathbf{K}] - \zeta_m[\mathbf{M}]\}\{\mathbf{y}_m\} = 0, \tag{1}$$

where $[\mathbf{K}]$, $[\mathbf{C}]$, and $[\mathbf{M}]$ are stiffness, damping and mass matrices, respectively. Also $\{\mathbf{y}_m\}$ is the mth eigenvector of the system and ζ_m is its corresponding eigenvalue related to the systems natural frequencies defined as

$$\zeta_m = (2\pi f_m)^2. \tag{2}$$

2.1. Derivatives of eigenvalues with respect to design variables

The differentiation of Eq. (1) with respect to a design variables b results in

$$\{[\mathbf{K}]' - \zeta'_m[\mathbf{M}] - \zeta_m[\mathbf{M}]'\}\{\mathbf{y}_m\} + \{[\mathbf{K}] - \zeta_m[\mathbf{M}]\}\{\mathbf{y}'_m\} = 0. \tag{3}$$

Pre-multiplication of Eq. (3) by $\{\mathbf{y}_m\}^{\mathrm{T}}$, the second term will take the form $(\{\mathbf{y}_m\}^{\mathrm{T}} \cdot \{[\mathbf{K}] - \zeta_m[\mathbf{M}]\})\{\mathbf{y}'_m\}$ which will be zero, because the term in parentheses is actually the transpose of Eq. (1) which is zero. Re-arrangement of the terms results in

$$\zeta'_m = \{\mathbf{y}_m\}^{\mathrm{T}}\{[\mathbf{K}]' - \zeta_m[\mathbf{M}]'\}\{\mathbf{y}_m\}, \tag{4}$$

where $()' = \partial/\partial b$. Eigenvectors are normalized with respect to the mass matrix as

$$\{\mathbf{y}_i\}^{\mathrm{T}}[\mathbf{M}]\{\mathbf{y}_j\} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{5}$$

In general where there may be k structural parameters used for dynamic optimization, Eq. (3) may be written in the partial form as

$$\frac{\partial \zeta_m}{\partial b_j} = \{\mathbf{y}_m\}^{\mathrm{T}} \left(\frac{\partial [\mathbf{K}]}{\partial b_j} - \zeta_m \frac{\partial [\mathbf{M}]}{\partial b_j} \right) \{\mathbf{y}_m\}, \quad j = 1, \ldots, k. \tag{6}$$

Differentiating Eq. (4) with respect to the parameters b and rearranging gives

$$\begin{aligned} &\zeta_m'' + (\{\mathbf{y}_m\}^{\mathrm{T}}[\mathbf{M}]'\{\mathbf{y}_m\})\zeta_m' + (\{\mathbf{y}_m\}^{\mathrm{T}}[\mathbf{M}]''\{\mathbf{y}_m\} + 2\{\mathbf{y}_m\}^{\mathrm{T}}[\mathbf{M}]'\{\mathbf{y}_m'\})\zeta_m \\ &\quad - 2\{\mathbf{y}_m\}^{\mathrm{T}}[\mathbf{K}]'\{\mathbf{y}_m'\} - \{\mathbf{y}_m\}^{\mathrm{T}}[\mathbf{K}]''\{\mathbf{y}_m\} = 0. \end{aligned} \tag{7}$$

It is therefore necessary to calculate $\{\mathbf{y}_m'\}$.

2.2. Derivative of eigenvectors with respect to design variables

Pre-multiplying Eq. (4) by $[\mathbf{M}]^{-1}$ and assuming $[\mathbf{D}] = [\mathbf{M}]^{-1}[\mathbf{K}]$ gives

$$([\mathbf{D}] - \zeta_m[\mathbf{I}])\{\mathbf{y}_m\}' + ([\mathbf{M}]^{-1}[\mathbf{K}]' - \zeta_m'[\mathbf{I}] - \zeta_m[\mathbf{M}]^{-1}[\mathbf{M}]')\{\mathbf{y}_m\} = \{\mathbf{0}\}, \tag{8}$$

where $[\mathbf{I}]$ is the diagonal unit matrix. It is noted that from Eq. (1) one can write

$$([\mathbf{D}] - \zeta_m[\mathbf{I}])\{\mathbf{y}_m\} = \{\mathbf{0}\}. \tag{9}$$

Eq. (9) implies that matrix $[\mathbf{D}]$ has the same eigenvalues and eigenvectors of the system. To establish the derivative of mass normalized mode shapes with respect to design parameters b, $\{\mathbf{y}_m\}'$ is expanded in the space of eigenvectors:

$$\{\mathbf{y}_m\}' = \sum_{i=1}^{i=n} \lambda_i \{\mathbf{y}_i\}, \tag{10}$$

where n is the number of mode shapes and λ_i's are the corresponding coefficients to be calculated. Substituting Eq. (10) into Eq. (8) and noting that $[\mathbf{D}]\{\mathbf{y}_m\} = \zeta_m\{\mathbf{y}_m\}$, one has

$$\sum_{i=1}^{i=n} \{\lambda_i(\zeta_i - \zeta_m)\{\mathbf{y}_i\}\} + ([\mathbf{M}]^{-1}[\mathbf{K}]' - \zeta_m'[\mathbf{I}] - \zeta_m[\mathbf{M}]^{-1}[\mathbf{M}]')\{\mathbf{y}_m\} = \{0\}. \tag{11}$$

Pre-multiplying (9) by $\{\mathbf{y}_i\}^{\mathrm{T}}[\mathbf{M}]$, one arrives at

$$\lambda_i = \frac{\{\mathbf{y}_i\}^{\mathrm{T}}([\mathbf{K}]' - \zeta_m[\mathbf{M}]')\{\mathbf{y}_m\}}{(\zeta_m - \zeta_i)}, \quad i \neq m. \tag{12}$$

Finally, to calculate λ_m, which is the projection of the derivative of the vector on itself, one differentiates Eq. (5) with respect to design parameters b and also use Eq. (10) to arrive at

$$\lambda_m = \frac{-1}{2}\{\mathbf{y}_m\}^{\mathrm{T}}[\mathbf{M}]'\{\mathbf{y}_m\}. \tag{13}$$

The case of coalescent eigenvalues is excluded in this work, although the method can be extended to cater for that case as well. In the case of repeated eigenvalues, problems arise as any linear combination of eigenvectors corresponding to the repeated eigenvalue is also a valid eigenvector and the eigenvectors of the repeated eigenvalues are not unique. In the case of repeated eigenvalues, the eigenderivatives are obtained through the solution of another subeigen problem [9–11]. In addition, Eq. (12) is singular for the repeated eigenvalue. Throughout the paper, matrix and vector quantities are enclosed in [ ] and { } brackets, respectively, and barred quantities denote initial values of the variables in their pre-modified state.

3. Approximate solution based on second order differential equation of eigenvalues

The second differentiation of Eq. (4) with respect to design variables gave rise to terms containing second derivative of eigenvalues and first derivative of the eigenvectors. The resulting equation is a second order differential equation:

$$\zeta_m'' + A\zeta_m' + B\zeta_m + C = 0, \tag{14a}$$

$$A = (\{\mathbf{y}_m\}^{\mathrm{T}}[\mathbf{M}]'\{\mathbf{y}_m\}), \quad B = (\{\mathbf{y}_m\}^{\mathrm{T}}[\mathbf{M}]''\{\mathbf{y}_m\} + 2\{\mathbf{y}_m\}^{\mathrm{T}}[\mathbf{M}]'\{\mathbf{y}_m'\}),$$

$$C = -2\{\mathbf{y}_m\}^{\mathrm{T}}[\mathbf{K}]'\{\mathbf{y}_m'\} - \{\mathbf{y}_m\}^{\mathrm{T}}[\mathbf{K}]''\{\mathbf{y}_m\}. \tag{14b}$$

The initial conditions of Eq. (14a) are

$$\zeta_m|_{b=\bar{b}} = \bar{\zeta}_m, \quad \zeta_m'|_{b=\bar{b}} = \{\{\mathbf{y}_m\}^{\mathrm{T}}([\mathbf{K}]' - \zeta_m[\mathbf{M}]')\{\mathbf{y}_m\}\}|_{b=\bar{b}} = \bar{\zeta}_m'. \tag{14c}$$

As an approximation, the coefficients of the differential equation (14a) are assumed to be constants. This implies again that the eigenvectors remain constant with respect to changes in the design variables. The solution of differential equation (14) results in ζ_m as a function of b. However, it is noted that all the coefficients of the equation vary with b. Therefore, the solution will offer a good approximation for dynamical characteristics of structure in terms of b in the vicinity of $\bar{b}$. The result will be in the form $\zeta_m = \zeta_m(b)$, which will be again solved for b to determine a required frequency shift $\Delta\zeta_m$:

$$\Delta\zeta_m = \zeta_m(b) - \bar{\zeta}_m. \tag{15}$$

Eq. (15) is generally a non-linear equation and can be solved using an iterative scheme. It is possible that the optimization problem could contain several design variables. This situation will result in a system of second order partial differential equations with independent variables b_j:

$$\zeta_m'' + A_j\zeta_m' + B_j\zeta_m + C_j = 0, \quad j = 1, \ldots, k, \tag{16}$$

where k is the number of design variables. The above system of differential equations whose coefficients were defined in Eq. (14b) should be solved together with their corresponding initial conditions. In order to reduce Eq. (16) and simplify its solution, additional practical equations of constraint may be introduced.

3.1. *Method 1: proportionality constraint for the second order system of differential equations*

To reduce the number of unknowns and hence the number of differential equations in (16), it is possible to define simple relations between design variables based on a practical judgement. As an example, all the design variables may be linearly related through a global variable α as

$$\frac{(b_j - \bar{b}_j)}{\bar{b}_j} = \alpha\omega_j, \quad j = 1, \ldots, k, \tag{17}$$

where ω_j are optional weights and may be chosen using engineering judgement. Therefore, having α as the only independent variable, the resulting differential equation from Eq. (7) for eigenvalue ζ_m will be

$$\frac{\partial^2\zeta_m}{\partial\alpha^2} + \left(\{\mathbf{y}_m\}^{\mathrm{T}}\frac{\partial[\mathbf{M}]}{\partial\alpha}\{\mathbf{y}_m\}\right)\frac{\partial\zeta_m}{\partial\alpha} + \left(\{\mathbf{y}_m\}^{\mathrm{T}}\frac{\partial^2[\mathbf{M}]}{\partial\alpha^2}\{\mathbf{y}_m\} + 2\{\mathbf{y}_m\}^{\mathrm{T}}\frac{\partial[\mathbf{M}]}{\partial\alpha}\frac{\partial\{\mathbf{y}_m\}}{\partial\alpha}\right)\zeta_m - 2\{\mathbf{y}_m\}^{\mathrm{T}}\frac{\partial[\mathbf{K}]}{\partial\alpha}\frac{\partial\{\mathbf{y}_m\}}{\partial\alpha} - \{\mathbf{y}_m\}^{\mathrm{T}}\frac{\partial^2[\mathbf{K}]}{\partial\alpha^2}\{\mathbf{y}_m\} = 0 \tag{18a}$$

with initial conditions:

$$\zeta_m|_{\alpha=0} = \bar{\zeta}_m; \quad \zeta'_m|_{\alpha=0} = \left\{\{\mathbf{y}_m\}^{\mathrm{T}}\left(\frac{\partial[\mathbf{K}]}{\partial\alpha} - \zeta_m\frac{\partial[\mathbf{M}]}{\partial\alpha}\right)\{\mathbf{y}_m\}\right\}\Bigg|_{\alpha=0} = \bar{\zeta}'_m. \tag{18b}$$

Also, for every scalar, vector, or matrix quantity X, one has

$$\frac{\partial X}{\partial\alpha} = \sum_{j=1}^{j=k}\frac{\partial X}{\partial b_j}\frac{\partial b_j}{\partial\alpha} = \sum_{j=1}^{j=k}\frac{\partial X}{\partial b_j}(\omega_j\bar{b}_j), \tag{19}$$

$$\frac{\partial^2 X}{\partial\alpha^2} = \sum_{j=1}^{j=k}\frac{\partial^2 X}{\partial b_j^2}(\omega_j\bar{b}_j)^2. \tag{20}$$

4. Second order partial derivative of eigenvalues

In addition to the second derivative of ζ_m with respect to a design variable, it is also required to establish the second order partial derivative of ζ_m with respect to two different variables. Therefore, differentiating Eq. (1) with respect to design variables b_i and b_j, one can write

$$\frac{\partial^2\zeta_m}{\partial b_i\,\partial b_j} = \{\mathbf{y}_m\}^{\mathrm{T}}\left(\frac{\partial^2[\mathbf{K}]}{\partial b_i\,\partial b_j} - \frac{\partial\zeta_m}{\partial b_j}\frac{\partial[\mathbf{M}]}{\partial b_i} - \zeta_m\frac{\partial^2[\mathbf{M}]}{\partial b_i\,\partial b_j}\right)\{\mathbf{y}_m\} + 2\{\mathbf{y}_m\}^{\mathrm{T}}\left(\frac{\partial[\mathbf{K}]}{\partial b_i} - \zeta_m\frac{\partial[\mathbf{M}]}{\partial b_i}\right)\frac{\partial\{\mathbf{y}_m\}}{\partial b_j} \tag{21a}$$

or

$$([\mathbf{K}] - \zeta_m[\mathbf{M}])\frac{\partial^2\{\mathbf{y}_m\}}{\partial b_i\,\partial b_j} + \left(\frac{\partial^2[\mathbf{K}]}{\partial b_i\,\partial b_j} - \frac{\partial^2\zeta_m}{\partial b_i\,\partial b_j}[\mathbf{M}] - \frac{\partial\zeta_m}{\partial b_i}\frac{\partial[\mathbf{M}]}{\partial b_j} - \frac{\partial\zeta_m}{\partial b_j}\frac{\partial[\mathbf{M}]}{\partial b_i} - \zeta_m\frac{\partial^2[\mathbf{M}]}{\partial b_i\,\partial b_j}\right)\{\mathbf{y}_m\} + \left(\frac{\partial[\mathbf{K}]}{\partial b_i} - \frac{\partial\zeta_m}{\partial b_i}[\mathbf{M}] - \zeta_m\frac{\partial[\mathbf{M}]}{\partial b_i}\right)\frac{\partial\{\mathbf{y}_m\}}{\partial b_j} + \left(\frac{\partial[\mathbf{K}]}{\partial b_j} - \frac{\partial\zeta_m}{\partial b_j}[\mathbf{M}] - \zeta_m\frac{\partial[\mathbf{M}]}{\partial b_j}\right)\frac{\partial\{\mathbf{y}_m\}}{\partial b_i} = 0. \tag{21b}$$

It is noted from differentiation of Eq. (1) with respect to every b that

$$\left(\frac{\partial[\mathbf{K}]}{\partial b} - \frac{\partial\zeta_m}{\partial b}[\mathbf{M}] - \zeta_m\frac{\partial[\mathbf{M}]}{\partial b}\right)\{\mathbf{y}_m\} = -([\mathbf{K}] - \zeta_m[\mathbf{M}])\frac{\partial\{\mathbf{y}_m\}}{\partial b}. \tag{22}$$

Pre-multiplying Eq. (21b) by $\{\mathbf{y}_m\}^{\mathrm{T}}$, one arrives at

$$\frac{\partial^2 \zeta_m}{\partial b_i\, \partial b_j} = \{\mathbf{y}_m\}^{\mathrm{T}}\left(\frac{\partial^2[\mathbf{K}]}{\partial b_i\, \partial b_j} - \frac{\partial \zeta_m}{\partial b_i}\frac{\partial[\mathbf{M}]}{\partial b_j} - \frac{\partial \zeta_m}{\partial b_j}\frac{\partial[\mathbf{M}]}{\partial b_i} - \zeta_m \frac{\partial^2[\mathbf{M}]}{\partial b_i\, \partial b_j}\right)\{\mathbf{y}_m\} - 2\,\frac{\partial\{\mathbf{y}_m\}^{\mathrm{T}}}{\partial b_i}([\mathbf{K}] - \zeta_m[\mathbf{M}])\,\frac{\partial\{\mathbf{y}_m\}}{\partial b_j}. \tag{23a}$$

Substitution of Eqs. (10) and (12) into Eq. (23a) yields

$$\frac{\partial^2 \zeta_m}{\partial b_i\, \partial b_j} = \{\mathbf{y}_m\}^{\mathrm{T}}\left(\frac{\partial^2[\mathbf{K}]}{\partial b_i\, \partial b_j} - \frac{\partial \zeta_m}{\partial b_i}\frac{\partial[\mathbf{M}]}{\partial b_j} - \frac{\partial \zeta_m}{\partial b_j}\frac{\partial[\mathbf{M}]}{\partial b_i} - \zeta_m \frac{\partial^2[\mathbf{M}]}{\partial b_i\, \partial b_j}\right)\{\mathbf{y}_m\} + 2\sum_{k=1}^{k=n}(\lambda_{mik}\lambda_{mjk}(\zeta_m - \zeta_k)). \tag{23b}$$

λ_{mrk} is the component of the vector $\partial\{\mathbf{y}_m\}/\partial b_r$ on $\{\mathbf{y}_k\}$ which is defined as

$$\frac{\partial\{\mathbf{y}_m\}}{\partial b_r} = \sum_{k=1}^{k=n}(\lambda_{mrk}\{\mathbf{y}_k\}), \tag{24a}$$

$$\lambda_{mrk} = \frac{\{\mathbf{y}_k\}^{\mathrm{T}}\left(\dfrac{\partial[\mathbf{K}]}{\partial b_r} - \zeta_m \dfrac{\partial[\mathbf{M}]}{\partial b_r}\right)\{\mathbf{y}_m\}}{\zeta_m - \zeta_k}, \quad k \neq m, \tag{24b}$$

$$\lambda_{mrm} = \frac{-1}{2}\{\mathbf{y}_m\}^{\mathrm{T}}\frac{\partial[\mathbf{M}]}{\partial b_r}\{\mathbf{y}_m\}, \quad k = m. \tag{24c}$$

As it is seen, the second derivative of an eigenvalue requires the derivative of eigenvectors. Lin et al. [6], have proposed a practical perturbed form to compute the eigenvector derivatives, requiring only the eigenvector itself and the inverse of mass and stiffness matrices. Also, Alvin has introduced an iterative method with improved accuracy and efficiency for the eigenderivative computations. For the case of repeated eigenvalues where two or more eigenvalues are equal and Eq. (24b) is singular, the methods outlined in Refs. [9–11] can be adopted to obtain the eigenvector derivatives. For the special case of two repeated eigenvalues, the method outlined in the references shows how to compute the eigenvector derivative.

5. Total differential form for the second order approximation

The second order total differential form of $F(x_i)$ as a function of several variables x_i can be expressed as

$$\mathrm{d}F = \sum_{i=1}^{i=k}\left\{\frac{\partial F}{\partial x_i}\mathrm{d}x_i\right\} + \sum_{i=1}^{i=k}\sum_{j=1}^{j=k}\left\{\frac{1}{2}\left(\frac{\partial^2 F}{\partial x_i\, \partial x_j}\right)\mathrm{d}x_i\, \mathrm{d}x_j\right\}, \tag{25}$$

where d is the total differential operator. As a second order approximation, Eq. (25), will be used for the cases where finite variation in the variables and functions are involved.

5.1. Method 2: unconstrained method

Here, Eq. (25) will be used as a second order approximation for eigenvalue optimization. This is equivalent to the use of second order expansion of Taylor series for multi-variable functions. From Eq. (25) one has

$$\Delta\zeta_m = \sum_{i=1}^{i=k} \left\{ \frac{\partial \zeta_m}{\partial b_i} \Delta b_i \right\} + \sum_{i=1}^{i=k} \sum_{j=1}^{j=k} \left\{ \frac{1}{2} \left(\frac{\partial^2 \zeta_m}{\partial b_i \, \partial b_j} \right) \Delta b_i \, \Delta b_j \right\}. \tag{26}$$

Obviously, the number of parameters to be changed is more than the number of frequencies to be shifted. To have a square system of equations without any constrains, additional frequency changes $\Delta\zeta_j$ must be introduced to the system. Therefore, Eq. (26) was set up to have the same number of frequency changes $\Delta\zeta_j$ as the number of modified design variables, with the superfluous frequency shifts set to zero:

$$\sum_{i=1}^{i=k} \sum_{j=1}^{j=k} \left\{ \frac{1}{2} \left(\frac{\partial^2 \zeta_m}{\partial b_i \, \partial b_j} \right) \Delta b_i \, \Delta b_j \right\} + \sum_{i=1}^{i=k} \left\{ \frac{\partial \zeta_m}{\partial b_i} \Delta b_i \right\} - \Delta\zeta_m = 0, \quad m = 1, \ldots, k, \tag{27}$$

where $\partial^2\zeta_m/(\partial b_i \, \partial b_j)$ can be obtained from Eq. (23).

Eq. (27) offers a system of non-linear equations, which may be solved by the Newton–Raphson iterative method using the following scheme:

$$\{\Delta\mathbf{b}^{(i+1)}\} = \{\Delta\mathbf{b}^{(i)}\} - [\mathbf{J}^{(i)}]^{-1}\{\mathbf{F}^{(i)}\}, \tag{28}$$

where the right superscript shows the number of iterations and $\{\Delta\mathbf{b}\}$ is the vector of unknowns. Also, $\{\mathbf{F}\}$ is vector of equations with the components F_m defined as

$$F_m = \sum_{i=1}^{i=k} \sum_{j=1}^{j=k} \left\{ \frac{1}{2} \left(\frac{\partial^2 \zeta_m}{\partial b_i \, \partial b_j} \right) \Delta b_i \, \Delta b_j \right\} + \sum_{i=1}^{i=k} \left\{ \frac{\partial \zeta_m}{\partial b_i} \Delta b_i \right\} - \Delta\zeta_m. \tag{29}$$

The Jacobian matrix $[\mathbf{J}] = \left[\frac{\partial F}{\partial \Delta b}\right]$ is defined as

$$J_{ij} = \frac{\partial F_i}{\partial \Delta b_j} = \sum_{k=1}^{k=n} \left\{ \left(\frac{\partial^2 \zeta_i}{\partial b_j \, \partial b_k} \right) \Delta b_k \right\} + \frac{\partial \zeta_i}{\partial b_j}. \tag{30}$$

It is much more efficient to conduct the optimization by including only the design variables which are most sensitive to a change in ζ_m. This is ensured by choosing the variables with maximum $\partial\zeta_i/\partial b_j$ obtained from Eq. (6). In order to apply Newton–Raphson scheme, the results Δb_j of the linear part of the system were used as our initial vector to begin the iteration process:

$$\sum_{i=1}^{i=k} \left\{ \frac{\partial \zeta_j}{\partial b_i} \Delta b_i \right\} = \Delta\zeta_j, \quad j = 1, \ldots, k. \tag{31}$$

Therefore, writing Eq. (30) in matrix form gives

$$\{\Delta\mathbf{b}^{(0)}\} = [\mathbf{S}]^{-1} \cdot \{\Delta\boldsymbol{\zeta}\}, \tag{32}$$

where

$$S_{ij} = \partial\zeta_i/\partial b_j. \tag{33}$$

5.2. Method 3: constrained method based on optimization of an objective function

In Section 5.1, additional frequency changes were introduced in order to obtain a square system of equations. It is also possible to achieve a square system of equations by defining physical constraints. These constraints may be based on practical engineering considerations. Alternatively, it is possible to define objective functions for optimization purposes. As an example, if the function to be optimized is the mass of the structure, Eq. (26) should be solved such that the added weight to the structure Δm is a minimum. Consider that added mass to the structure is defined as

$$\Delta m = m(b_1, b_2, \ldots, b_k). \tag{34}$$

Adopting a Lagrange multipliers method for the problem, the functional to be minimized is

$$\Pi = m(b_1, \ldots, b_k) + \lambda\left\{\sum_{i=1}^{i=k}\sum_{j=1}^{j=k}\left\{\frac{1}{2}\left(\frac{\partial^2\zeta_m}{\partial b_i\,\partial b_j}\right)\Delta b_i\,\Delta b_j\right\} + \sum_{i=1}^{i=k}\left\{\frac{\partial\zeta_m}{\partial b_i}\Delta b_i\right\} - \Delta\zeta_m\right\}, \tag{35}$$

where λ is the Lagrange multiplier. The minimization is carried out according to the equations

$$\frac{\partial\Pi}{\partial b_j} = \frac{\partial\Pi}{\partial\lambda} = 0, \quad j = 1, \ldots, k \tag{36}$$

or

$$\begin{cases} \dfrac{\partial m(b_1, \ldots, b_k)}{\partial b_j} + \lambda\left\{\sum_{i=1}^{i=k}\left\{\left(\dfrac{\partial^2\zeta_m}{\partial b_i\,\partial b_j}\right)\Delta b_i\right\} + \dfrac{\partial\zeta_m}{\partial b_j}\right\} = 0, \quad j = 1, \ldots, k, \\ \sum_{i=1}^{i=k}\sum_{j=1}^{j=k}\left\{\dfrac{1}{2}\left(\dfrac{\partial^2\zeta_m}{\partial b_i\,\partial b_j}\right)\Delta b_i\,\Delta b_j\right\} + \sum_{i=1}^{i=k}\left\{\dfrac{\partial\zeta_m}{\partial b_i}\Delta b_i\right\} - \Delta\zeta_m = 0. \end{cases} \tag{37}$$

The above is a non-linear system of equations and the Newton–Raphson iterative solution is again used for their non-linear solution. Similar to Eq. (32), the results from the linear part of the system, which is obtained by ignoring the higher order terms in Eq. (37), may be used as the initial value for the iterative procedure.

5.3. Method 4: constrained method based on proportionality

In many engineering problems it is not feasible to introduce additional physical constraints in order to achieve sufficient number of equations. Instead, it may be possible to choose additional criterion to reduce the number of unknowns for a particular kind of problem. As an example, consider the problem where it is required to vary b_j by changing the thickness or cross sectional area of several finite elements in order to optimize ζ_m. It is possible to consider a global variable α

with respect to which all the design variables change proportional to a weight ω_j as

$$\frac{(b_j - \bar{b}_j)}{\bar{b}_j} = \alpha\omega_j, \quad j = 1, \ldots, k. \tag{38}$$

This reduces the problem to a single unknown variable α. The weights can be chosen from a sensitivity analysis of the design variables. Eqs. (19), (20), (23), (26), and (38) will therefore result in a single variable parabolic equation to be solved for α:

$$\alpha^2\left\{\sum_{i=1}^{i=k}\sum_{j=1}^{j=k}\left(\tfrac{1}{2}\omega_i\omega_j\bar{b}_i\bar{b}_j\frac{\partial^2\zeta_m}{\partial b_i\,\partial b_j}\right)\right\} + \alpha\left\{\sum_{i=1}^{i=k}\left(\frac{\partial\zeta_m}{\partial b_i}(\omega_i\bar{b}_i)\right)\right\} - \Delta\zeta_m = 0. \tag{39}$$

Also, $\partial^2\zeta_1/(\partial b_i\,\partial b_j)$ will be obtained from Eq. (23) to establish Eq. (39). It should be noted that for practical purposes, for example in a plate or shell structures, the additional required change in thickness of an element can be converted into its equivalent stiffeners which will be attached to the structure at their corresponding elemental position.

5.4. Method 5: partially constrained method based on proportionality

In Section 5.1, the number of unknowns was reduced to one, by using a global variable α. It is possible to simultaneously reduce both the number of unknowns and increase the number of equations to achieve a square system of equation. This can be considered as the combination of Methods 2 and 4, as explained previously. Therefore, by adopting p independent new design variables, α_r, where $p<k$, and p optimized eigenvalue, ζ_r, the constraints can be written as

$$\frac{(b_i - \bar{b}_i)}{\bar{b}_i} = \sum_{s=1}^{s=p}\alpha_s\omega_{si}, \quad i = 1, \ldots, k \tag{40}$$

and the system of equations will become

$$\Delta\zeta_r = \sum_{i=1}^{i=p}\sum_{j=1}^{j=p}\left\{\frac{1}{2}\left(\frac{\partial^2\zeta_r}{\partial\alpha_i\,\partial\alpha_j}\right)\alpha_i\alpha_j\right\} + \sum_{i=1}^{i=p}\left\{\frac{\partial\zeta_r}{\partial\alpha_i}\alpha_i\right\}, \quad r = 1, \ldots, p. \tag{41}$$

To compute the derivatives with respect to new variables, one has

$$\frac{\partial\zeta_r}{\partial\alpha_s} = \sum_{i=1}^{i=k}\frac{\partial\zeta_r}{\partial b_i}\frac{\partial b_i}{\partial\alpha_s} = \sum_{i=1}^{i=k}\frac{\partial\zeta_r}{\partial b_i}(\omega_{si}\bar{b}_i), \quad r, s = 1, \ldots, p \tag{42}$$

and

$$\frac{\partial^2\zeta_r}{\partial\alpha_s\,\partial\alpha_t} = \frac{\partial}{\partial\alpha_t}\left(\sum_{m=1}^{m=k}\frac{\partial\zeta_r}{\partial b_m}(\omega_{sm}\bar{b}_m)\right) = \sum_{m=1}^{m=k}\sum_{n=1}^{n=k}\left\{\frac{\partial^2\zeta_r}{\partial b_m\,\partial b_n}(\omega_{sm}\omega_{tn}\bar{b}_m\bar{b}_n)\right\}, \tag{43}$$

where p is number of unknowns, α_s, and k is the number of possible material or geometric design variables to be changed. Also, ω_{sj} is the weight used for the jth design variable corresponding to the sth independent variable. Again, decision on the magnitude of weights ω_{sj} may be made upon engineering judgement, structural feasibility, and other considerations. Substitution of Eqs. (42)

and (43) into Eq. (41) yields

$$\Delta\zeta_r = \sum_{i=1}^{i=p}\sum_{j=1}^{j=p}\left\{\frac{1}{2}\left(\sum_{m=1}^{m=k}\sum_{n=1}^{n=k}\left\{\frac{\partial^2\zeta_r}{\partial b_m\,\partial b_n}(\omega_{im}\omega_{jn}\bar{b}_m\bar{b}_n)\right\}\right)\alpha_i\alpha_j\right\} + \sum_{i=1}^{i=p}\left\{\sum_{m=1}^{m=k}\frac{\partial\zeta_r}{\partial b_m}(\omega_{im}\bar{b}_m)\alpha_i\right\}, \quad r = 1,\ldots,p \tag{44}$$

or

$$\Delta\zeta_r = \frac{1}{2}\sum_{i=1}^{i=p}\sum_{j=1}^{j=p}\sum_{m=1}^{m=k}\sum_{n=1}^{n=k}\left(\omega_{im}\omega_{jn}\bar{b}_m\bar{b}_n\frac{\partial^2\zeta_r}{\partial b_m\,\partial b_n}\alpha_i\alpha_j\right) + \sum_{i=1}^{i=p}\sum_{m=1}^{m=k}\left(\omega_{im}\bar{b}_m\frac{\partial\zeta_r}{\partial b_m}\alpha_i\right), \quad r = 1,\ldots,p. \tag{45}$$

Obviously, Eq. (45) is a system of non-linear equations, which may be solved through a Newton–Raphson iterative scheme, similar to the solution used for Eq. (27).

6. Case studies

To show the accuracy and efficiency of the presented second order methods, the same examples presented in Part I of this paper are considered. The results are compared against their exact solutions, which are obtained by computing the eigenvalues of the modified optimum structure using the ANSYS program. These eigenvalues are then compared with the eigenvalues used as inputs into the inverse algorithm.

Example 1. Consider the same simply supported 2-D truss structure shown in Fig. 1 presented in Part I. The objective is to shift the first frequency of the structure with a second order approximation. Sensitivity analysis reveals the most sensitive members for the first eigenvalue shift. The most sensitive members are then changed:

(a) with equal proportions using differential equations (Method 1);
(b) with equal proportions using Method 4.

The ANSYS program is used to find the dynamic characteristics of the structure. The 5 natural frequencies of the structure are given in Table 1. Table 2 shows the most sensitive members with

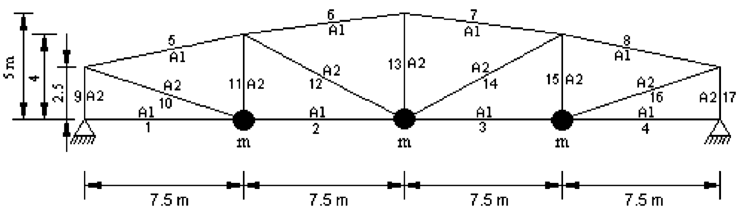

Fig. 1. 2-D truss with concentrated masses: $A_1 = 250$ cm^2, $A_2 = 82.5$ cm^2, $m = 5000$ kg, $E = 2.1E11$ N/m^2.

Table 1
Dynamic characteristics of structure (Fig. 1)

i	Mode number i				
	1	2	3	4	5
Eigenvalue ζ_i	15790	56707	127098	333202	1218025
Frequency f_i (Hz)	20.0	37.9	56.7	91.9	175.7

Table 2
First eigenvalue sensitivity (Fig. 1)

	j								
	1	2	5	6	9	10	11	12	13
A_j	250	250	250	250	82.5	82.5	82.5	82.5	82.5
$\mathrm{d}\zeta_1/\mathrm{d}A_j$	11961.0	11961.0	49322.1	57963.2	40490.1	500171	0.2	6442.3	24579.5

Table 3
Values of $\partial^2\zeta_1/(\partial b_i \partial b_j)$ (Fig. 1)

j	i					
	5	6	7	8	10	16
5	−314266	51925	51907	206376	−3186738	2092696
6	51925	−272703	−272703	51907	526532	526349
7	51907	−272703	−272703	51925	526349	526532
8	206376	51907	51925	−314266	2092696	−3186738
10	−3186738	526532	526349	2092696	−32314326	21220353
16	2092696	526349	526532	−3186738	21220353	−32314326

highest $\mathrm{d}\zeta_1/\mathrm{d}A_i$. In all the formulations, only the first five modal shapes, obtained from ANSYS are used.

(a) From Table 3, elements 5, 6, 10, and symmetrically 7, 8, and 16 are selected for modification such that in all cases

$$\frac{(A_i - \bar{A}_i)}{\bar{A}_i} = \alpha, \quad i = 5, 6, 7, 8, 10, 16, \tag{46}$$

where it is assumed that $\omega_i = 1$. For every truss element number, i, the stiffness matrix $[\mathbf{K}^{(i)}]$ is defined as

$$[\mathbf{K}^{(i)}] = \frac{E_i A_i}{L_i} \begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}$$

from which their derivatives are found. Since the mass matrix is constant, from (14) one has

$$\zeta_1'' + A\zeta_1' + B\zeta_1 + C = 0,$$

where $A = B = 0$ and

$$C = -2\{\mathbf{y}_1\}^{\mathrm{T}} \frac{\partial[\mathbf{K}]}{\partial\alpha}\frac{\partial\{\mathbf{y}_1\}}{\partial\alpha} = -2\{\mathbf{y}_1\}^{\mathrm{T}} \left(\sum_i \bar{A}_i \frac{\partial[\mathbf{K}]}{\partial A_i}\right)\left(\sum_i \bar{A}_i \frac{\partial\{\mathbf{y}_1\}}{\partial A_i}\right) = 2101.$$

The initial conditions are

$$\zeta_1|_{\alpha=0} = 15790, \quad \zeta_1'|_{\alpha=0} = \{\mathbf{y}_1\}^{\mathrm{T}}\left(\frac{\partial[\mathbf{K}]}{\partial\alpha}\right)\{\mathbf{y}_1\}\bigg|_{\alpha=0} = 13617.$$

The solution therefore will be

$$\zeta_1 = -1051\alpha^2 + 13617\alpha + 157909. \tag{47}$$

Figs. 2 and 3 give the comparison between the solution obtained from the ANSYS computer program and Eq. 47 and the error in frequency shift respectively. Due to the closeness of the results from the exact and the approximate solution, the results from the two methods in Fig. 2 are not distinguishable. It can be seen from the above results that even for large structural changes, Eq. (47) offers a very accurate frequency shift model. Comparing with the first order solution, the maximum error has decreased from 2.7 to 1.15 which is quite significant.

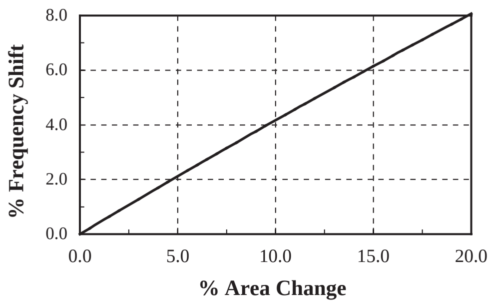

Fig. 2. Method 1 for frequency shift (Fig. 1).

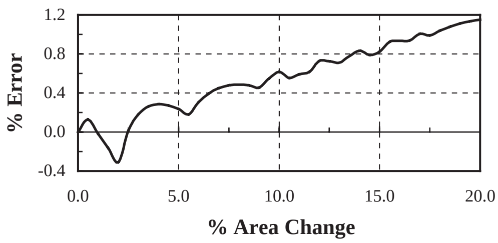

Fig. 3. Percentage error of Method 1 (Fig. 1).

(b) Again using Eq. (46) as the proportionality constraint, since the weights are unity $\omega_i = 1$, from Eq. (39) one has

$$\Delta\zeta_m = \alpha^2 \sum_i \sum_j \left(\frac{1}{2} \bar{b}_i \bar{b}_j \frac{\partial^2 \zeta_m}{\partial b_i \, \partial b_j} \right) + \alpha \sum_i \left(\bar{b}_i \frac{\partial \zeta_m}{\partial b_i} \right), \tag{48}$$

where from Eqs. (23) and (24) one has

$$\frac{\partial^2 \zeta_m}{\partial b_i \, \partial b_j} = -2 \frac{\partial \{\mathbf{y}_m\}^{\mathrm{T}}}{\partial b_i} [\mathbf{K}] \frac{\partial \{\mathbf{y}_m\}}{\partial b_j} = 2 \sum_{k=1}^{k=n} (\lambda_{mik} \lambda_{mjk} (\zeta_m - \zeta_k)), \tag{49}$$

$$\lambda_{mik} = \frac{1}{\zeta_m - \zeta_k} \{\mathbf{y}_k\}^{\mathrm{T}} \frac{\partial [\mathbf{K}]}{\partial b_i} \{\mathbf{y}_m\}, \quad k \neq m, \tag{50a}$$

$$\lambda_{mim} = \frac{-1}{2} \{\mathbf{y}_m\}^{\mathrm{T}} \frac{\partial [\mathbf{M}]}{\partial b_i} \{\mathbf{y}_m\}. \tag{50b}$$

Here, $m = 1$, $n = 5$, and $i, j = 5, 6, 7, 8, 10, 16$. Table 3 shows the magnitude of $\partial^2 \zeta_1 / (\partial b_i \partial b_j)$. It is noted that for $i = j$, it is possible to compute $\partial^2 \zeta_1 / \partial b_i^2$ using Eq. (7):

$$\frac{\partial^2 \zeta_1}{\partial b_i^2} = 2 \{\mathbf{y}_1\}^{\mathrm{T}} \frac{\partial [\mathbf{K}]}{\partial b_i} \left\{ \frac{\partial \mathbf{y}_1}{\partial b_i} \right\}. \tag{51}$$

This may slightly differ from the results computed using Eq. (49), unless all the eigenvectors and eigenvalues of the structure are taken into account.

Therefore, using Tables 2 and 3 together with Eq. (48),

$$\zeta_1 = -1051\alpha^2 + 13617\alpha + 15790 \tag{52}$$

is arrived at. It is interesting to note that the resulting Eq. (52) is exactly the same as Eq. (47) obtained using the other method. Again, the percentage of frequency shift and the accuracy of (52) are shown in Figs. 4 and 5, respectively. Again comparison with the first order solution reveals that the maximum error has decreased from about 2.7 to about 1.15.

Example 2. The first frequency of the same 2-D truss structure presented in Example 2 of Part I of this paper is again shown in Fig. 6 to be optimised. The consistent mass matrix used for the truss

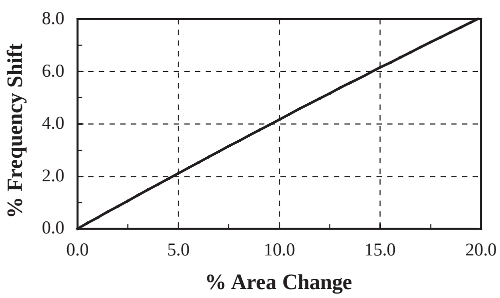

Fig. 4. Frequency shift vs. %area change (Fig. 1).

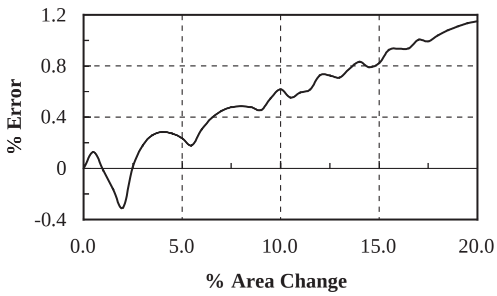

Fig. 5. Percentage error in frequency shift (Fig. 1).

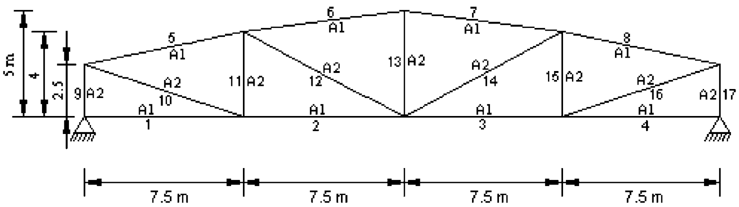

Fig. 6. 2-D truss with consistent mass: $A_1 = 250\ \text{cm}^2$, $A_2 = 100\ \text{cm}^2$, $\rho = 2880$ kg, $E = 2.0\text{E}11\ \text{N/m}^2$.

members is given by

$$[\mathbf{M}^{(i)}] = \frac{\rho_i A_i L_i}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}.$$

$\partial[\mathbf{M}]/\partial A_i$ is initially computed for the truss members. The dynamic characteristics and sensitivity are given in Tables 4 and 5. In order to shift the first frequency of the truss, the properties of the most sensitive members are changed by

(a) using Method 1 with equal absolute weights for all the chosen members;
(b) using Method 4 with equal absolute weights for all the chosen members.

(a) Now consider that elements 2, 3, 10, 12, 13, 14, and 16 are selected for modification in order to achieve an increase in the first eigenfrequency of the structure. Assuming equal percentages for their area changes, one has

$$\frac{A_i - \bar{A}_i}{\bar{A}_i} = \omega_i \alpha, \quad i = 2, 3, 10, 12, 13, 14, 16, \tag{53}$$

where $\omega_i = 1$ ($i = 10, 16$), and $\omega_i = -1$ ($i = 2, 3, 12, 13, 14$). The sign of the chosen weights depend on whether the members have an increasing or a decreasing effect on the frequency shift.

Table 4
Dynamic characteristics (Fig. 8)

	Mode i				
	1	2	3	4	5
ζ_i	15796	83311	135637	270344	886489
f_i (Hz)	20.0	45.9	58.6	82.8	149.9

Table 5
First eigenvalue sensitivity (Fig. 8)

	j								
	1	2	5	6	9	10	11	12	13
A_j	250	250	250	250	100	100	100	100	100
$d\zeta_1/dA_j$	−10813	−77940	27647	−25032	34463	347683	−33112	−98508	−70429

Therefore, from Eqs. (18)–(20) one has

$$\zeta_1'' + A\zeta_1' + B\zeta_1 + C = 0, \tag{54a}$$

$$A = \{\mathbf{y}_1\}^{\mathrm{T}}[\mathbf{M}]'\{\mathbf{y}_1\}, \quad B = 2\{\mathbf{y}_1\}^{\mathrm{T}}[\mathbf{M}]'\{\mathbf{y}_1'\}, \quad C = -2\{\mathbf{y}_1\}^{\mathrm{T}}[\mathbf{K}]'\{\mathbf{y}_1'\}, \tag{54b}$$

$$\zeta_1|_{\alpha=0} = \bar{\zeta}_1, \quad \zeta_1'|_{\alpha=0} = \{\mathbf{y}_1\}^{\mathrm{T}}\{[\mathbf{K}]' - \zeta_1[\mathbf{M}]'\}\{\mathbf{y}_1\}|_{\alpha=0} = \bar{\zeta}'|_{\alpha=0}, \tag{54c}$$

where all the differentiations are carried out with respect to α. To compute the coefficients and the initial conditions of Eq. (54), Eqs. (19) and (20) are used. The resulting differential equation is

$$\zeta_1'' - 0.290\zeta_1' - 0.143\zeta_1 - 347 = 0, \tag{55a}$$

$$\zeta_1|_{\alpha=0} = 15796, \quad \zeta_1'|_{\alpha=0} = 13525 \tag{55b}$$

and the solution is

$$\zeta_1 = 65258 - 68543 \times e^{-0.091\alpha} + 19081 \times e^{0.382\alpha}. \tag{56}$$

Figs. 7 and 8 show the frequency shift and its error against the percent of area change in the structural members. It can be seen that the error is quite small and negligible. It is seen that for a 20% cross-sectional area change, a shift of 8.5% in the frequency is achieved with an induced error of about 0.5%, compared to the error of the first order method which was about 2.2%.

(b) Again assuming

$$\frac{(A_i - \bar{A}_i)}{\bar{A}_i} = \omega_i\alpha, \quad i = 2, 3, 10, 12, 13, 14, 16 \tag{57}$$

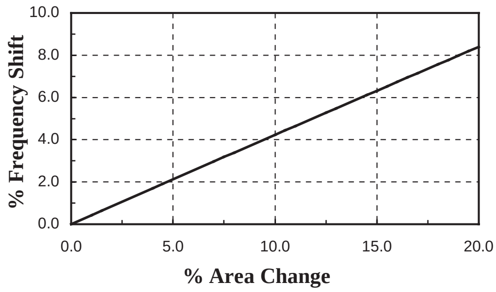

Fig. 7. Frequency shift vs. %area change (Fig. 6).

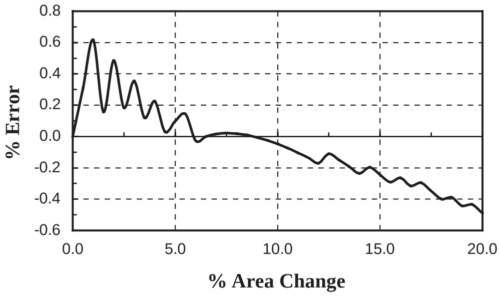

Fig. 8. Percentage error in frequency shift (Fig. 6).

with $\omega_i = 1$ ($i = 10, 16$), and $\omega_i = -1$ ($i = 2, 3, 12, 13, 14$), from Eq. (39) one has

$$\alpha^2\left\{\sum_{i=1}^{i=k}\sum_{j=1}^{j=k}\left(\tfrac{1}{2}\,\omega_i\omega_j\bar{b}_i\bar{b}_j\,\frac{\partial^2\zeta_m}{\partial b_i\,\partial b_j}\right)\right\} + \alpha\left\{\sum_{i=1}^{i=k}\left(\frac{\partial\zeta_m}{\partial b_i}(\omega_i\bar{b}_i)\right)\right\} - \Delta\zeta_m = 0,$$

where

$$\frac{\partial^2\zeta_m}{\partial b_i\,\partial b_j} = \{\mathbf{y}_m\}^{\mathrm{T}}\left(-\frac{\partial\zeta_m}{\partial b_i}\frac{\partial[\mathbf{M}]}{\partial b_j} - \frac{\partial\zeta_m}{\partial b_j}\frac{\partial[\mathbf{M}]}{\partial b_i}\right)\{\mathbf{y}_m\} + 2\sum_{k=1}^{k=n}(\lambda_{mik}\lambda_{mjk}(\zeta_m - \zeta_k)), \tag{58a}$$

$$\lambda_{mrk} = \begin{cases} \dfrac{\{\mathbf{y}_k\}^{\mathrm{T}}\left(\dfrac{\partial[\mathbf{K}]}{\partial b_r} - \zeta_m\dfrac{\partial[\mathbf{M}]}{\partial b_r}\right)\{\mathbf{y}_m\}}{\zeta_m - \zeta_k}, & k \neq m, \\ -\tfrac{1}{2}\{\mathbf{y}_m\}^{\mathrm{T}}\left(\dfrac{\partial[\mathbf{M}]}{\partial b_r}\right)\{\mathbf{y}_m\}, & k = m. \end{cases} \tag{58b}$$

Table 6 shows the values of $\partial^2\zeta_1/\partial b_i\partial b_j$ for the selected members. Eqs. (57) and (58) result in the following relation for the eigenvalue change:

$$\zeta_1 = 2541\alpha^2 + 13525\alpha + 15796. \tag{59}$$

Table 6
$\partial^2\zeta_1/(\partial b_i \partial b_j)$

j	i 2	3	10	12	13	14	16
2	794341	896510	−1870307	1085340	768256	1085340	−1870307
3	896510	794341	−1870307	1085340	768256	1085340	−1870307
10	−1870307	−1870307	−32233433	−2150844	−1503353	−2150844	−1153658
12	1085340	1085340	−2150844	−134826	927843	1311400	−2150844
13	768256	768256	−1503353	927843	373959	927843	−1503353
14	1085340	1085340	−2150844	1311400	927843	−134826	−2150844
16	−1870307	−1870307	−1153658	−2150844	−1503353	−2150844	−32233433

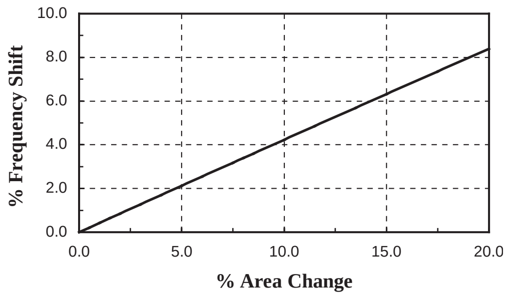

Fig. 9. Frequency shift vs. %area change (Fig. 6).

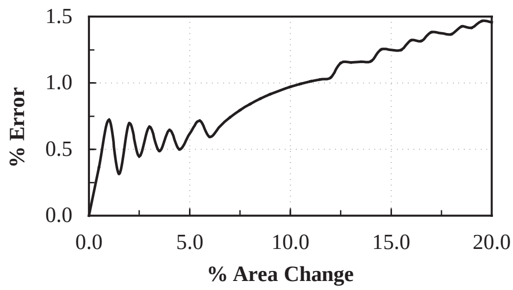

Fig. 10. Percentage error in frequency shift (Fig. 6).

The result of which are shown in Figs. 9 and 10. As it is seen, for 20% cross-sectional area change, an 8.5% frequency shift is achieved with an induced error of about 1.5% which is less than the first order error which was about 2.1%.

Example 3. Consider the same 2-D arch shown in Fig. 11 presented in Example 3 of Part I of this paper. The initial thickness of the arch is 30 cm and the material and geometrical properties are given in the figure. The consistent mass matrix is used for the elements in the model. The objective

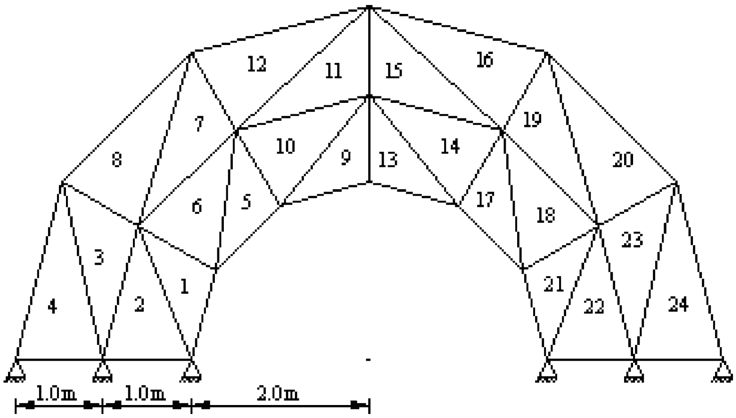

Fig. 11. A 2-D plane stress arch structure: $E = 1E10\ \mathrm{N/m^2}$, $\rho = 2500\ \mathrm{kg/m^3}$, $v = 0.25$, $t = 30$ cm.

Table 7
Natural frequencies

	Mode i					
	1	2	3	4	5	6
ζ_i	198590	388383	1023320	1372854	2801953	3738760
f_i (Hz)	70.93	99.19	161.00	186.48	266.41	307.74

again is to initially identify here the most sensitive elements with respect to the first frequency. This will be then followed by modification of the properties of those elements by

(a) using the differential equation method (Method 1);
(b) using the proportionality constraints (Method 4).

The first few frequencies are given in Table 7. Also, Tables 8 and 9 show the sensitivity of the first two frequencies with respect to members' thicknesses.

(a) Adopting equal weights in the members' relative change of thickness, and with negative signs for members with negative $\mathrm{d}\zeta_1/\mathrm{d}t_i$, one has

$$\frac{(t_i - \bar{t}_i)}{\bar{t}_i} = \omega_i \alpha, \quad i = 2, 4, 11, 12, 15, 16, 22, 24, \tag{60}$$

where $\omega_i = 1$ ($i = 2, 4, 22, 24$), and $\omega_i = -1$ ($i = 11, 12, 15, 16$). Therefore, from Eqs. (13) and (14) one arrives at

$$\begin{cases} \zeta'' - 0.336\zeta' - 0.131\zeta + 1359 = 0, \\ \zeta|_{\alpha=0} = 198590, \quad \zeta'|_{\alpha=0} = 146301, \end{cases} \tag{61}$$

Table 8
First eigenvalue sensitivity

	Member i							
	2	4	11	12	15	16	22	24
$d\zeta_1/dt_i$	61376	81799	−44391	−56270	−44391	−56270	61376	81799

Table 9
Second eigenvalue sensitivity

	Member i							
	1	3	11	12	15	16	21	23
$d\zeta_2/dt_i$	149351	61969	−109910	−96791	−109910	−96791	149351	61969

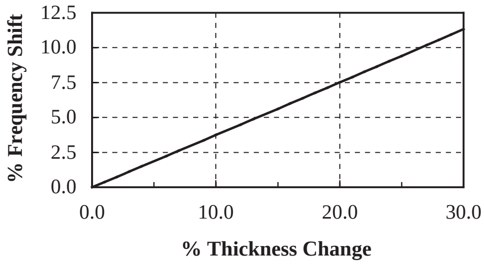

Fig. 12. Frequency shift vs. %area change (Fig. 11).

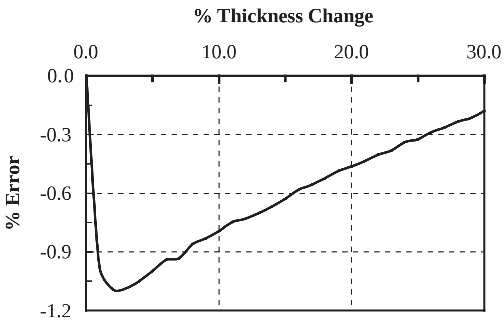

Fig. 13. Percentage error in frequency shift (Fig. 11).

which gives rise to an equation for eigenvalue change in terms of proportionality factor α:

$$\zeta = 10355 - 49537e^{-0.231\alpha} + 237773e^{0.567\alpha}. \tag{62}$$

The variation of frequency shift and its error are shown in Figs. 12 and 13 respectively.

It is observed that even considering the initial part of the graph, the induced error with this method is 1.1% which is remarkably lower than the first order method with an error of 2.7%.

Table 10
$\partial^2\zeta_1/(\partial b_i \partial b_j)$ for sensitive elements using the first six modes

j	i							
	2	4	11	12	15	16	22	24
2	−37516	−40807	−14024	−22761	−16529	−23809	9300	12840
4	−40807	−64972	−11805	−19464	−22602	−26945	12839	22495
11	−14024	−11805	16763	23191	26097	31975	−16529	−22603
12	−22761	−19464	23191	29627	31975	38013	−23810	−26945
15	−16529	−22602	26097	31975	16764	23191	−14024	−11804
16	−23809	−26945	31975	38013	23191	29627	−22761	−19463
22	9300	12839	−16529	−23810	−14024	−22761	−37516	−40806
24	12840	22495	−22603	−26945	−11804	−19463	−40806	−64974

(b) Again,

$$\frac{t_i - \bar{t}_i}{\bar{t}_i} = \omega_i \alpha, \quad i = 2, 4, 11, 12, 15, 16, 22, 24, \tag{63}$$

where $\omega_i = 1$ ($i = 2, 4, 22, 24$) and $\omega_i = -1$ ($i = 11, 12, 15, 16$). It is now necessary to establish Eq. (39) for the problem. Therefore,

$$\alpha^2 \sum_{i=1}^{i=k} \sum_{j=1}^{j=k} \left(\tfrac{1}{2} \omega_i \omega_j \bar{b}_i \bar{b}_j \frac{\partial^2 \zeta_1}{\partial b_i \, \partial b_j} \right) + \alpha \sum_{i=1}^{i=k} \left(\frac{\partial \zeta_1}{\partial b_i} (\omega_i \bar{b}_i) \right) - \Delta\zeta_1 = 0, \tag{64}$$

where according to Eq. (45),

$$\frac{\partial^2 \zeta_1}{\partial b_i \, \partial b_j} = -\{\mathbf{y}_1\}^{\mathrm{T}} \left(\frac{\partial \zeta_1}{\partial b_i} \frac{\partial [\mathbf{M}]}{\partial b_j} + \frac{\partial \zeta_1}{\partial b_j} \frac{\partial [\mathbf{M}]}{\partial b_i} \right) \{\mathbf{y}_1\} - 2 \frac{\partial \{\mathbf{y}_1\}^{\mathrm{T}}}{\partial b_i} ([\mathbf{K}] - \zeta_1 [\mathbf{M}]) \frac{\partial \{\mathbf{y}_1\}}{\partial b_j}. \tag{65}$$

The magnitude of $\partial^2\zeta_1/(\partial b_i \partial b_j)$, which are obtained using the first six natural modes, are shown in Table 10. The resulting approximate equation for eigenvalue in the vicinity of the current configuration of the structure will therefore be

$$\zeta = 36440\alpha^2 + 146301\alpha + 198591. \tag{66}$$

The exact change of frequency is the same as that shown in Fig. 12. As shown in Fig. 14, the percentage of error induced due to the application of this method is 1.1% which is much less than the 7.2% obtained by its corresponding first order method.

Therefore, with a 30% change in the cross-sectional area of eight most sensitive elements up to 30%, the frequency will shift more than 11% with an error of less than 0.7% which shows a good accuracy for the method.

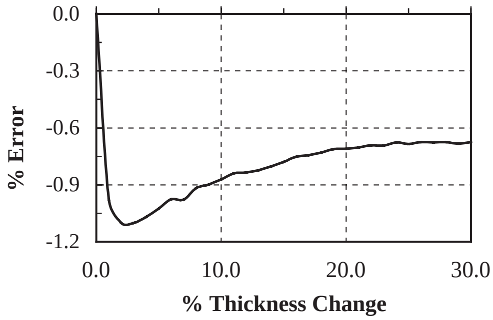

Fig. 14. Percentage error in frequency shift using Method 4 (Fig. 11).

7. Conclusions

In this work, some second order methods are introduced to relocate the desired natural frequencies of the structures. This is done through the second order Taylor's expansion series resulting in second order differential or binomial equations. The method is then combined with objective functions to incorporate design constraints. The formulations are quite general and applicable to all finite element structures since all the stiffness and mass matrices derivatives are obtained regardless of the type of the element. If the mass or stiffness matrix has a complicated formulation with respect to a design variable, its derivative can be obtained numerically. The accuracy of the proposed methods is tested by conducting several case studies. The results show very significant improvements when compared with the first order approaches presented in part one of this paper.

Acknowledgements

The research work leading to the results presented in this paper was jointly funded by the EPSRC and the Astrium company whose support is greatly appreciated.

References

[1] R.L. Fox, M.P. Kapoor, Rates of change of eigenvalues and eigenvectors, *American Institute of Aeronautics and Astronautics Journal* 6 (12) (1968) 2426–2429.

[2] R.B. Nelson, Simplified calculation of eigenvector derivatives, *American Institute of Aeronautics and Astronautics Journal* 14 (9) (1976) 1201–1205.

[3] B.P. Wang, Improved approximate methods for computing eigenvector derivatives in structural dynamics, *American Institute of Aeronautics and Astronautics Journal* 29 (6) (1991) 1018–1020.

[4] O. Zhang, A. Zerva, Iterative method for calculating derivatives of eigenvectors, *American Institute of Aeronautics and Astronautics Journal* 34 (5) (1996) 1088–1090.

[5] J.-G.Beliveau, S. Cogan, G. Lallement, F. Ayer, Iterative least-squares calculation for modal eigenvector sensitivity, *American Institute of Aeronautics and Astronautics Journal* 34 (2) (1996) 385–391.

[6] R.M. Lin, Z. Wang, M.K. Lim, A practical algorithm for efficient computation of eigenvector sensitivities, *Computer Methods in Applied Mechanics and Engineering* 130 (1996) 355–367.
[7] R.M. Lin, M.K. Lim, Structural sensitivity analysis via reduced-order analytical mode, *Computer Methods in Applied Mechanics and Engineering* 121 (1995) 345–359.
[8] K.F. Alvin, Efficient computation of eigenvector sensitivities in structural dynamics, *American Institute of Aeronautics and Astronautics Journal* 35 (11) (1997) 1760–1766.
[9] S.P. Kenny, G.J.W. Hou, Approximate analysis for repeated eigenvalue problems with application to controls–structures integrated design, Nasa Technical Paper 3439, June 1994.
[10] W.C. Mills-Curran, Calculation of eigenvector derivatives for structures with repeated eigenvalues, *American Institute of Aeronautics and Astronautics Journal* 26 (7) (1988) 867–871.
[11] R.L. Dailey, Eigenvectors derivatives with repeated eigenvalues, *American Institute of Aeronautics and Astronautics Journal* 27 (4) (1989) 486–491.